NEMO Application Round 2007-08

Application I: Analyzing Numbers in a "Fun" Way (40 Points)

There are many ways to look at numbers. You are probably familiar with "factoring," where you can express any integer as a product of powers of primes. For example, $36 = (2^2)(3^2)$. In a more formal sense, factoring means that you are expressing a number n as the product $p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots p_k^{e_k}$ where all the p's are prime and all the e's are positive integers.

Let's define a new type of factoring, called *fun factoring*. This type of factoring applies to all rational numbers $q > 0$. We first choose a prime number p. We then write q as $p^a \cdot (r/s)$ where a is an integer, r and s are integers not divisible by p, and r and s share no factor other than 1. The expression $p^a \cdot (r/s)$ is called the *fun factorization of* q *with respect to* p. The value p^{-a} is called the *happiness of* q *with respect to* p. For example, let $q = 12/25$ and let $p = 3$. The fun factorization of q with respect to 3 is $3^1 \cdot (4/25)$, that is, $p = 3$, $a = 1$, $r = 4$, and $s = 25$. The happiness of q with respect to 3 is $3^{-1} = 1/3$. Also note that the fun factorization of q with respect to 7 is $7^0 \cdot (12/25)$, and the happiness with respect to 7 is $7^{-0} = 1$.

1. **(4 Points)** Find the fun factorization of q with respect to p for the following pairs of q and p: (a) $q = 123$, $p = 3$, (b) $q = 41/256$, $p = 2$, (c) $q = 111/125$, $p = 5$, and (d) $q = 255/131$, $p = 7$. In other words, give p, a, r, and s.
2. **(2 Points)** Find the happiness of q with respect to p for the pairs of q and p given in the previous problem.
3. **(4 Points)** A number is called *ideally happy* if its happiness with respect to *every* prime p is 1. How many rational numbers are ideally happy? Either list them all or give a general rule.
4. **(6 Points)** A number is called *ecstatic* if its happiness with respect to *every* prime p is at least 1.
   (a) Are there any ecstatic numbers? If so, how can you tell whether a rational number is ecstatic?
   (b) Are all ideally happy numbers also ecstatic?
   (c) Are there any numbers greater than 1 that are ecstatic?
5. **(4 Points)** A number is *depressed* if its happiness with respect to *any* prime p is at most 1. Are there any depressed numbers? If so, how can you identify a depressed number?
6. **(6 Points)** Suppose $d \neq 1$ is a depressed number and $e \neq 1$ is an ecstatic number. Can the product de be (a) ideally happy? (b) Depressed but not ideally happy? (e) Ecstatic but not ideally happy? If you answered "yes" to any of (a)–(c), then give examples of d and e to support your assertion. If you answered "no," discuss why.
7. **(4 Points)** Melissa and Alan are both thinking of a rational number. Melissa says, "The happiness of my number with respect to 5 is 1/25." Alan, appalled, says, "That's true about my number as well!" Are Melissa and Alan necessarily thinking of the same number?
8. **(6 Points)** Jonathan and Shriram are both thinking of a rational number as well, with both the numerator and denominator divisible by no prime factor greater than 1000. These two boys have a lot of time on their hands, so they calculate the happiness of their numbers with respect to every prime number less than 1000. They find their numbers have equal happiness with respect to every such prime. Are their numbers necessarily equal?
9. **(4 Points)** Amanda says, "I'm thinking of a number whose happiness with respect to 2 is 1/36." Janelle blurts out, "That's not possible." Is Janelle correct, or is she just being pretentious?

Application II: Mendel's Laws of Genetics (40 Points)

In the mid 1800's, the Austrian monk Gregor Mendel experimented with inheritance in pea plants to define three basic laws of genetics: (1) the law of dominance, (2) the law of segregation, and (3) the law of independent assortment. For every trait (ie, height of a plant, color of a seed, etc.), all living beings have at least two alleles. Each allele codes for a specific representation of that trait—for color, one allele might code for red seeds and the other for green seeds. The law of dominance states that when one allele is dominant over another allele (in this case, say that red is dominant), an organism will show the characteristic of the dominant allele (in this case, the red seeds) if at least one of its two alleles is dominant (red). In short, the law of segregation states that a child will inherit one random allele from each parent. Finally, the law of independent assortment says that one trait will not affect another; for example, tall plants are not necessarily blue. If an allele is not dominant, then it is called recessive. (For the purposes of this problem, we will ignore codominance, etc.)

Conventional notation will be used in this problem. A capital letter will be used to represent a dominant allele, and a corresponding lowercase letter will be used to represent the recessive allele. The letter used will be the first letter of name of the dominant allele.

1. **(4 Points)** In pea plants, the tall allele (T) is dominant over the short allele (t). If a purebred tall plant (both alleles are T) is crossed with a purebred short plant (both alleles are t), (a) what is the probability that the first offspring is tall? (b) What is the probability that the first offspring is short?

2. **(4 Points)** Now assume that two heterozygous pea plants (Tt) are crossed. What is the probability that at least two out of four offspring are tall?

3. **(6 Points)** What is the probability that a short pea plant has at least one tall parent, if each parent is equally likely to be either TT, Tt, or tt?

   For problems 4 and 5, consider the following: In a certain alien race, each child has three parents. For each of the child's traits, two parents are randomly chosen, and the child inherits one allele from each. In this race, short (S) is dominant over tall (s).

4. **(6 Points)** Find (a) the probability that the child is short, given that the parents are SS, Ss, and ss; and (b) the probability that the child is tall given the parents are Ss, Ss, and ss.

5. **(10 Points)** Find the probability that a tall alien has at least one tall parent, if each parent is equally likely to be either SS, Ss, or ss.

6. **(10 Points)** In a certain group of pea plants, each pea plant has an $x < \frac{1}{2}$ chance of being short (tt), an x chance of being tall (TT), and a $1-2x$ chance of being heterozygous (Tt). If all offspring of these plants have the same probability distribution (x chance of tt, x of TT, $1-2x$ of Tt), find x.