NEMO Application Round 2007-08: Solutions

Application I: Analyzing Numbers in a "Fun" Way (40 Points)

1. The following are the fun factorizations required:

   (a) $123 = 3^1 \cdot (41/1)$

   (b) $41/256 = 2^{-8} \cdot (41/1)$

   (c) $111 = 5^{-3} \cdot (111/1)$

   (d) $255/111 = 7^0 \cdot (255/111)$

2. The happiness for each is (a) $3^{-1} = 1/3$, (b) $2^8 = 256$, (c) $5^3 = 125$, and (d) $7^0 = 1$.

3. The only way a number can have a happiness rating of 1 with respect to a prime number p is if the fun factorization has p^0. Since ideally happy numbers have happiness with respect to p equal to 1 always, regardless of which p we choose, the fun factorization will have p^0 in it. Note that since r and s are not divisible by p, p^a is the highest power of p that divides the number. Thus, for an ideally happy number, the highest power of *any* prime that divides it is $p^0 = 1$. Thus, an ideally happy number is not divisible by any power of any prime, which implies that 1 is the only ideally happy number.

4. We answer the question in three parts:

   (a) If a number q is ecstatic, then we know that $p^{-a} \geq 1$ for any p and a in a given fun factorization of q. Then, $a \leq 0$ for every fun factorization of q with respect to p. It follows that q, when put in lowest terms, must have 1 as a numerator. (If it had any other number as a numerator, we could find a prime factor that divides that number, and if we take the fun factorization of q with that prime, a would be greater than 0.)

   (b) An ideally happy number has a happiness of 1 with respect to any prime. It follows that it also has a happiness of *at least* 1 with respect to any prime as well, since 1 is at least 1. Thus, all ideally happy numbers are ecstatic.

   (c) If a number q is greater than 1, then its numerator when placed in lowest terms cannot possible be 1 (since then its denominator would be less than one, which is not possible). Thus, there are no ecstatic numbers less than 1.

5. Using the same logic as part (a) of the previous question, it follows that all depressed numbers, when placed in lowest terms, have a denominator equal to 1. In other words, depressed numbers are integers.

6. All of the cases are possible:

   (a) $d = 5$, $e = 1/5$, $de = 1$

   (b) $d = 25$, $e = 1/5$, $de = 5$

   (c) $d = 5$, $e = 1/25$, $de = 1/5$

7. Melissa and Alan need not think of the same number. Suppose Melissa is thinking of $25 = 5^2 \cdot (1/1)$ and Alan is thinking of $50 = 5^2 \cdot (2/1)$. Then, both numbers have the same happiness with respect to 5, but they are not equal.

8. Jonathan and Shriram are thinking of the same number. Note that if the happiness of q with respect to p is p^{-a}, then p^a divides q. Specifically, p^a is the highest power of p that will divide q. We can thus write $q = p^a \cdot k$ for some k not divisible by p. If we find the happiness of q with respect to all possible p, we can successfully make $k = 1$. In essence, we know that if two numbers have the same happiness with respect to all possible primes, then they have the same prime factorization, which implies that they are equal.

9. The happiness of any number with respect to p is in the form p^k for some integer k. In other words, it must be a power of the prime. Since $1/36$ is not a power of 2, Janelle is correct. Amanda must have made a miscalculation.

Application II: Mendel's Laws of Genetics

1. Since one parent (TT) will always donate a tall allele, the offspring will always be tall. Thus, the probability is 1.

2. The probability that both parents contribute a short allele (i.e., the child is short) is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. As such, there is a $\frac{3}{4}$ chance of a child being tall. If two of the four children are tall, we have 6 ways to choose the pair times $\frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4}$, which equals $\frac{54}{256}$. If three of the four children are tall, we have 4 ways to choose the triple times $\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}$, yielding $\frac{108}{256}$. If four of the children are tall, we have probability $(\frac{3}{4})^4 = 81/256$. Thus, the total probability is $\frac{243}{256}$.

3. If the pea plant is short (tt), then there are four possible combinations of parents: Tt $\times$ Tt, Tt $\times$ tt, tt $\times$ Tt, and tt $\times$ tt. (Yes, the second and third possibilities are in fact different.) For each of these cases, there are 1, 2, and 4 equally likely ways to get tt, respectively. Only the first three cases have tall parents, so the probability of getting a tall parent is $\frac{1+2+2}{1+2+2+4} = \frac{5}{9}$.

4. (a) If the parent SS is represented, the child is automatically short by the Law of Dominance. There is a $\frac{2}{3}$ chance of this. The other $\frac{1}{3}$ chance makes probability $\frac{1}{2}$ (if and only if S is represented from Ss). Thus we have $\frac{2}{3} + \frac{1}{6} = \frac{5}{6}$.

   (b) If Ss and Ss are chosen (probability $\frac{1}{3}$, then we have probability $\left(\frac{1}{2}\right)^2$ chance of being tall. If Ss and ss are chosen (probability $\frac{2}{3}$, we have a $\frac{1}{2}$ chance of being tall. Thus, there is a total chance of $\frac{1}{3} + \frac{1}{12} = \frac{5}{12}$ chance of being tall.

5. It may be helpful to make a table of all combinations that allow for a tall child:

$$\begin{array}{ccccccccc} SS & Ss & Ss & : & 1 & \times & 3 & = & 3 \\ Ss & Ss & Ss & : & 3 & \times & 1 & = & 3 \\ ss & ss & ss & : & 12 & \times & 1 & = & 12 \\ Ss & ss & ss & : & 8 & \times & 3 & = & 24 \\ SS & ss & ss & : & 4 & \times & 3 & = & 12 \\ Ss & Ss & ss & : & 5 & \times & 3 & = & 15 \\ SS & Ss & ss & : & 6 & \times & 2 & = & 12 \end{array}$$

   The first number represents the number of ways to get ss from that specific combination, while the next number is the number of permutations of that combination. We have a total of 81 ways to get ss from any parent combination. Only 6 of those ways feature no tall parents (the first two rows). So, the probability of a tall child having a tall parents is $\frac{81-18}{81} = \frac{7}{9}$.

6. To calculate the proportion of TT in the next generation, we look at all possibilites of parents: TT $\times$ TT, TT $\times$ Tt, Tt $\times$ TT, and Tt $\times$ Tt. In the first pair, the child is guaranteed to be TT. In the second two, the child has a $\frac{1}{2}$ chance, and in the last, a $\frac{1}{4}$. The probability of a parent being TT is x, just as the probability of a parent being Tt is $1 - 2x$. The probability of a child being TT is thus:

$$(x)^2(1) + 2(x)(1-2x)\left(\frac{1}{2}\right) + (1-2x)^2\left(\frac{1}{4}\right)$$

   This value must be equal to the proportion of TT in the next generation, which is x. Thus:

$$\begin{aligned} (x^2)(1) + 2(x)(1-2x)\left(\frac{1}{2}\right) + (1-2x)^2\left(\frac{1}{4}\right) &= x \\ x^2 + x(1-2x) + (1-2x)^2 &= x \\ 4x^2 + 4x(1-2x) + (1-2x)^2 &= 4x \\ -4x + 1 &= 0 \\ x &= \frac{1}{4} \end{aligned}$$