NEMO 2008 Problems - Individual Round

Solutions

1. Let C be the average cost per share. Then

$$C = \frac{100 \cdot 45 + 200 \cdot 51}{300} = \frac{45 + 2 \cdot 51}{3} = 15 + 2 \cdot 17 = \$49.00$$

2. The two dice can show any of (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) if the total number of pips is seven. There are 36 possible rolls, hence the probability is $\frac{6}{36} = \frac{1}{6}$.

3. We know that $5^0 = 1, 5^1 = 5, 5^2 = 25, 5^3 = 125, 5^4 = 625, 5^5 = 3125$. On Day 0 (Sunday) $5^0 = 1$ person is infected. On Day 1 (Monday) $5^0 + 5^1 = 6$ people are infected. In general, on Day n there are $5^0 + 5^1 + \cdots + 5^n$ people who are infected. $5^5 > 2600$, so everybody is infected by at most Day 5. However, on Day 4 we find that $\frac{5^5-1}{5-1} = \frac{3124}{4} = 781$ people are infected. Hence by Day 5 (Friday) the whole school is infected.

4.

$$\frac{17! \cdot 13! \cdot 7!}{16! \cdot 14! \cdot 5!} = \frac{17 \cdot 7 \cdot 6}{14} = 17 \cdot 3 = 51$$

5. If there are three boys, then the number of possible committees is $\binom{6}{3} \cdot \binom{5}{1}$. If there are two boys, then the number of possible committees is $\binom{6}{2} \cdot \binom{5}{2}$. If there is one boy, then the number of possible committees is $\binom{6}{1} \cdot \binom{5}{3}$. Hence the total number of committees is

$$\binom{6}{3} \cdot \binom{5}{1} + \binom{6}{2} \cdot \binom{5}{2} + \binom{6}{1} \cdot \binom{5}{3} = 20 \cdot 5 + 15 \cdot 10 + 6 \cdot 10 = 100 + 150 + 60 = 310$$

6. Let p be the amount of pizza Paul eats, and let m be the amount that Mary eats. We know that $m = \frac{1}{2}$. We also know that $p + \frac{1}{7} + \frac{1}{3} + \frac{1}{2} = 1$. But $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{5}{6} + \frac{1}{7} = \frac{41}{42}$. Hence $p = \frac{1}{42}$. The desired ratio is thus

$$\frac{m}{p} = \frac{\frac{1}{2}}{\frac{1}{42}} = 21$$

7. We group the sum as

$$0 + (-1 + 2) + (-3 + 4) + \cdots + (-99 + 100)$$

It is easy to see that each term in parentheses is equal to 1, and since there are 100 numbers in parentheses and 2 numbers per term in parentheses, there are 50 terms in parentheses. Thus the sum is equal to 50.

8. Let the triangle be ABC and let the line segment be drawn from A to M, the midpoint of BC. Since ABC is equilateral we have $AM \perp BC$ and $\angle ABC = 60°$. Hence AMB is a 30-60-90 triangle and $AM = \frac{s\sqrt{3}}{2}$. The desired ratio is thus

$$\frac{AM}{AB} = \frac{\frac{s\sqrt{3}}{2}}{s} = \frac{\sqrt{3}}{2}$$

9. We have $37^2 - 35^2 = (37 + 35)(37 - 35) = 72 \cdot 2 = 144 = 12^2$. Hence the hypotenuse has length 37. It follows that the triangle has area $\frac{\pi}{4} \cdot 37^2 = \frac{1369\pi}{4}$.

10. We have that the volume $V = s^3 = 6\sqrt{6}$. Similarly, the surface area $A = 6s^2 = 36$. A space diagonal has length $l_s = s\sqrt{3} = 3\sqrt{2}$, which can be found by double application of the Pythagorean Theorem. A face diagonal has length $l_f = s\sqrt{2} = 2\sqrt{3}$. Hence we have the sum

$$S = V + A + l_s + l_f = 36 + 3\sqrt{2} + 2\sqrt{3} + 6\sqrt{6}$$

11. Let the height of the cylinder be h and let its radius be r. Then the first condition gives us $2\pi rh = 2\pi$. Hence $rh = 1$. The second condition gives us

$$\begin{aligned} \pi r^2 - 4\pi r &= -4\pi \\ r^2 - 4r + 4 &= 0 \\ (r-2)^2 &= 0 \\ r &= 2 \end{aligned}$$

We have

$$V = \pi r^2 h = \pi (rh) \cdot r = \pi \cdot 1 \cdot 2 = 2\pi$$

12. Since the square base has area 16, its side length is 4. The four triangles that make up the sides thus have two sides of length 8 and one side of length 4. Label the vertices of the triangle such that $AB = AC = 8$ and $BC = 4$. Let M be the midpoint of BC. Since $\triangle ABC$ is isosceles, $AM \perp BC$. Hence by the Pythagorean Theorem, $AM = \sqrt{8^2 - 2^2} = 2\sqrt{15}$. Thus the area of $\triangle ABC$ is $4\sqrt{15}$. It follows that the total surface area of the pyramid is

$$S = 16 + 4 \cdot 4\sqrt{15} = 16 + 16\sqrt{15}$$

13. The sum S of the entire geometric series is given by

$$S = \frac{a_0}{1-r} = \frac{16}{1-\frac{1}{2}} = \frac{16}{\frac{1}{2}} = 32$$

Since the eleventh term is $16 \cdot \left(\frac{1}{2}\right)^{10} = \frac{1}{64}$, the sum of all the terms after the tenth one is $\frac{\frac{1}{64}}{\frac{1}{2}} = \frac{1}{32}$. Hence the tenth partial sum is $32 - \frac{1}{32}$, so the answer is 32.

14. $f(x) = x^3 + x^2 + 2x + 2 = (x+1)\left(x^2+2\right)$. Hence $f(-1) = 0$. The second term, however, cannot be zero when x is a real number because the square of any real numbers is nonnegative. Hence $f(x)$ has 1 real zero.

15. Let a, b, c be the sides of the triangle and let p, q, r be the radii of the circles. Since the circles are all externally tangent, we have $a = p + q, b = p + r, c = q + r$. Setting $p = 3, q = 5, r = 12$ gives $a = 8, b = 15, c = 17$. Thus the triangle is a right triangle, with area $\frac{1}{2}8 \cdot 15 = 60$.

16. Any integer divisible by 2 or 5 will not be counted. Hence the integers that are counted are 1, 3, 7, 9, 11, 13, 17, 19. Thus $\phi(20) = 8$.

17. $5 + \frac{6}{7} = \frac{41}{7}$. $3 + \frac{4}{\frac{41}{7}} = 3 + \frac{28}{41} = \frac{151}{41}$. $1 + \frac{2}{\frac{151}{41}} = 1 + \frac{82}{151} = \frac{233}{151}$.

18. If Alex pays with half-dollars, then he uses 33 half-dollars. But note that 5 dimes make half a dollar, as do 2 quarters. Since 2 and 5 have no nontrivial common divisors, the money paid in dimes must be a multiple of 50 cents, as must the money paid in quarters. There are 33+1 = 34 multiples of 50 cents between \$0.00 and \$16.50, including \$0.00. Hence there are 34 ways for Alex to pay.

19. The following positive integers divide 100: 1, 2, 4, 5, 10, 20, 25, 50, 100. Their sum is 217.

20. $4^b = 81$ implies $64^{\frac{b}{3}} = 81$. Hence $(3^a)^{\frac{b}{3}} = 3^{\frac{ab}{3}} = 81$. But $3^4 = 81$. Hence $\frac{ab}{3} = 4$, so $ab = 12$.

21. Let $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$. For this to have a limit, $\sqrt{1+x} = x$. Hence $x^2 - x - 1 = 0$ and $x = \frac{1 \pm \sqrt{5}}{2}$. But $x > 0$, hence $x = \frac{1+\sqrt{5}}{2}$.

22. Note that $a \diamond b = (a+1)(b+1) - 1$. Hence $2 \diamond 3 = 3 \cdot 4 - 1$ and $4 \diamond 5 = 5 \cdot 6 - 1$. Thus

$$(2 \diamond 3) \diamond (4 \diamond 5) = (3 \cdot 4) \cdot (5 \cdot 6) - 1 = 360 - 1 = 359$$

23. In base 10, a number is divisible by 9 if the sum of its digits is divisible by 9. The same holds in base 8: a number is divisible by 7 if the sum of its digits is divisible by 7. Note that 76 (base 8) $= 7 \cdot 8 + 6 = 62$. There are 8 numbers greater than 1 and less than 62 that are divisible by 7. Hence there are 8 vivacious integers in the given range.

24. Note that, because the radii of the inscribed circle are perpendicular to the sides, the area K of the triangle is given by $K = \frac{pr}{2}$ where p is the perimeter and r is the radius of the inscribed circle. $p = 9$, so $K = \frac{3}{2}\sqrt{15}$.

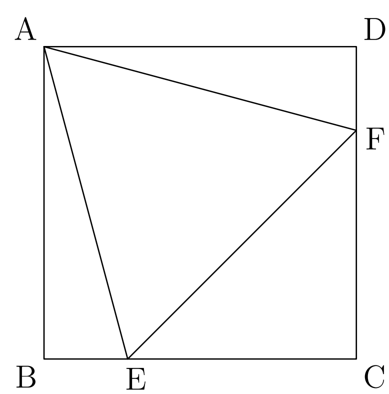
25. $\nabla 0 = 2^{2^0} + 1 = 2^1 + 1 = 3$. $\nabla 3 = 2^{2^3} + 1 = 2^8 + 1 = 257$. After checking all primes less than $\sqrt{257} < 17$, we find that 257 is prime. Hence it has one prime factor.

26. We have

$$\left(1 + \sqrt{2}\right)^4 = \left(3 + 2\sqrt{2}\right)^2 = 17 + 12\sqrt{2} = 17 + \sqrt{288} < 17 + \sqrt{289} = 17 + 17 = 34$$

27. Hyunji has to lose a cookie at least twice. Hence she can either lose twice and do nothing twice, or lose thrice and gain once. There are $\frac{4!}{2!}2! = 6$ permutations of the first path, and $\frac{4!}{3!1!} = 4$ permutations of the second path. The probability of the events on the first path is $\left(\frac{1}{2}\right)^2 \left(\frac{1}{4}\right)^2 = \frac{1}{64}$. The probability of the events on the second path is $\left(\frac{1}{4}\right)^3 \left(\frac{1}{4}\right) = \frac{1}{256}$. Hence the total probability of the first path is $\frac{6}{64}$ and the total probability of the second path is $\frac{4}{256} = \frac{1}{64}$. It follows that the probability that Hyunji exits with six cookies is $\frac{7}{64}$.

28. We label the points as shown.



Solution 1: Let the side length of $ABCD$ be s and let $\lambda = \frac{BE}{BC}$. Then $AE^2 = AF^2$, so $\frac{DF}{DC} = \lambda$. We have $AE^2 = AF^2 = \left(1+\lambda^2\right)s^2$ and $EF^2 = 2(1-\lambda)^2 s^2$. But $AE = AF = EF$, so $AE^2 = AF^2 = EF^2$. Hence

$$\begin{aligned} 1+\lambda^2 &= 2(1-\lambda)^2 \\ 1+\lambda^2 &= 2-4\lambda+2\lambda^2 \\ 0 &= 1-4\lambda+\lambda^2 \\ \lambda &= \frac{4\pm\sqrt{16-4}}{2} \\ \lambda &= 2\pm\sqrt{3} \end{aligned}$$

$0 < \lambda < 1$, hence $\lambda = 2-\sqrt{3}$. Let $[XYZ]$ denote the area of a polygon XYZ. We want to find $\frac{[AEF]}{[ABCD]}$. We know that $[ABCD] = s^2$. We also know that $[AEF] = [ABCD] - [ABE] - [ADF] - [ECF]$. We know that $[ABE] = [ADF] = \frac{1}{2}AB \cdot BE = \frac{1}{2}s \cdot \lambda s = \frac{2-\sqrt{3}}{2}s^2$. We also know that $[ECF] = \frac{1}{2}CE \cdot CF = \frac{1}{2}\left[(1-\lambda)s\right]^2 = \frac{1}{2}s^2\left(\sqrt{3}-1\right)^2 = \left(2-\sqrt{3}\right)s^2$. Hence $[AEF] = s^2 - 2\left(\frac{2-\sqrt{3}}{2}s^2\right) - \left(2-\sqrt{3}\right)s^2 = s^2\left(2\sqrt{3}-3\right)$. The desired ratio is thus

$$\frac{[AEF]}{[ABCD]} = \frac{s^2\left(2\sqrt{3}-3\right)}{s^2} = 2\sqrt{3}-3$$

Solution 2: Draw AC and let AC intersect EF at P. Note that $AP = AE \cdot \frac{\sqrt{3}}{2}$ and $PC = \frac{EF}{2}$. Hence $AC = s\sqrt{2} = AE \cdot \left(\frac{1+\sqrt{3}}{2}\right)$. Thus $\frac{AE^2}{s^2} = \left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^2 = \frac{8}{4+2\sqrt{3}} = \frac{4}{2+\sqrt{3}} = 8-4\sqrt{3}$. We have $[AEF] = \frac{AP \cdot EF}{2} = AE^2\frac{\sqrt{3}}{4}$. Also $[ABCD] = s^2$. Hence

$$\frac{[AEF]}{[ABCD]} = \frac{AE^2}{s^2} \cdot \frac{\sqrt{3}}{4} = \left(8-4\sqrt{3}\right) \cdot \frac{\sqrt{3}}{4} = 2\sqrt{3}-3$$

29. Note that for a quadratic of the form $x^2+px+q=0$, we can rewrite the equation in terms of the roots r_1, r_2 as $x^2-(r_1+r_2)x+r_1r_2=0$. Hence $r_1+r_2=-p$ and $r_1r_2=q$. But $(r_1+1)(r_2+1) = r_1r_2+r_1+r_2+1 = q-p+1$. In our given equation, after dividing through by the coefficient of the first term we have the equation $x^2 - \frac{37}{11}x + \frac{23}{11} = 0$. Hence

$$(a+1)(b+1) = \frac{23}{11} + \frac{37}{11} + 1 = \frac{71}{11}$$

30. Let r_n be the remainder of the n^{th} Fibonacci number F_n upon division by 5. $0 \leq r_n \leq 4$. Also, $r_{n+2} = r_{n+1} + r_n$ if $r_{n+1} + r_n \leq 4$ and $r_{n+2} = r_{n+1} + r_n - 5$ if $r_{n+1} + r_n \geq 5$. (For students familiar with modular arithmetic, $r_n = F_n \pmod{5}$). We know that $r_0 = r_1 = 1$. We can compute the r_n sequence fairly easily.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
r_n	2	3	0	3	3	1	4	0	4	4	3	2	0	2	2	4	1	0	1

Hence four numbers in the given set are divisible by 5.