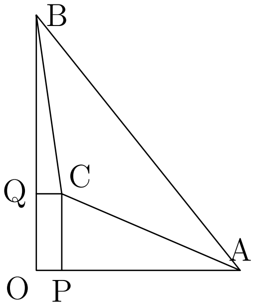
NEMO 2008 - Team Round
Solutions

1. Since $\frac{11}{2} > 5$, one of the rolls has to be at least a six. Thus it must be an eight. Now, the only way to generate a sum of more than eleven is to have one of the three possibilities $(5,8),(8,5),(8,8)$. There are $6*6=36$ possible ordered pairs of rolls. Hence the probability is
$$\frac{3}{36}=\frac{1}{12}$$

2. Solution 1: We know that if a triangle has vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$, then its area $K$ is given by the unsigned determinant
$$K = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$
Using either sub-determinants or expansion by minors, we find this determinant to be 46. Hence
$$K=23$$

Solution 2: For ease of computation, we translate the triangle right by two units and down by one unit. Then it has vertices $A(8,0),B(0,10),C(1,3)$. We let $P=(1,0),Q=(0,3)$, and let $O$ be the origin.



Let $[XYZ]$ denote the area of a polygon $XYZ$. Then $[ABC]=[ABO]-[APC]-[BQC]-[OPCQ]$. We have $[ABO]=\frac{1}{2}\cdot 8\cdot 10=40$. Similarly $[APC]=\frac{1}{2}\cdot 7\cdot 3=\frac{21}{2}$. We find $[BQC]=\frac{1}{2}\cdot 7\cdot 1=\frac{7}{2}$. Lastly, $[OPCQ]=1\cdot 3=3$. Hence
$$[ABC]=40-\frac{21}{2}-\frac{7}{2}-3=37-14=23$$

3. 2 cents must be paid in pennies. Thus the problem reduces to finding the number of ways to pay 45 cents with U.S. currency. If we use one quarter, then we must pay 20 cents more. This can be done according to the following table:

| Dimes | Nickels | Pennies |
|---|---|---|
| 2 | 0 | 0 |
| 1 | 2 | 0 |
| 1 | 1 | 5 |
| 1 | 0 | 10 |
| 0 | 4 | 0 |
| 0 | 3 | 5 |
| 0 | 2 | 10 |
| 0 | 1 | 15 |
| 0 | 0 | 20 |

Hence there are nine ways to make change with one quarter.

Now if we use no quarters, then we can use anywhere from zero to four dimes. But look at the table above. When we had 20 cents to pay using only nickels and dimes, we had five possibilities, because we could use anywhere from zero to four cents. We conclude that we can pay $5x$ cents in only nickels and dimes in exactly $x + 1$ ways.

| Dimes | Cents Left |
|---|---|
| 0 | 45 |
| 1 | 35 |
| 2 | 25 |
| 3 | 15 |
| 4 | 5 |

If we find the sum of $x + 1$ along the right-hand column, we get $2 + 4 + 6 + 8 + 10 = 30$ ways. Hence there are 39 ways in total.

Note: There is another technique (generating functions) that can be used to solve the problem in a slightly easier fashion, but the algebra involved is very complex.

4. Note that we can write the polynomial as $x^2(x-2)^2 + (x+1)^2$. Since squares of numbers are always nonnegative, the given expression can never be less than zero. Hence we need only find its zeros. The first expression is only zero when $x = 0$ and $x = 2$, and the second is only zero when $x = -1$. Since both expressions are nonnegative, there are no such integers.

5. For a sphere with radius $r$, the volume $V$ and surface area $S$ are given by

$$V = \frac{4\pi}{3}r^3 \qquad\qquad S = 4\pi r^2$$

Hence $\frac{V}{S} = \frac{r}{3}$. But we know that $\frac{V}{S} = 2$. Hence $\frac{r}{3} = 2$, so $r = 6$. This implies that

$$V = \frac{4\pi}{3} \cdot 6^3 = 8\pi \cdot 6^2 = 288\pi$$

6. If Alice is on either of the two end seats, there is only one seat next to her. If Alice is in any of the four center seats, there are two seats next to her. Hence there are ten possibilities for the arrangement of Alice and Fred. For each arrangement, there are 4! ways to order the

other four people. There are 6! ways to order all six people. Hence the desired probability $P$ is given by

$$P = \frac{10 \cdot 4!}{6!} = \frac{10}{5 \cdot 6} = \frac{1}{3}$$

7. We can count the number of ways, but this would take a long time. Thankfully, there is a better technique.

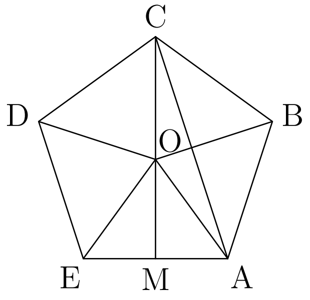
   Imagine that there are eight balls in a row, with walls on either end. We have two more walls that we will place among the eight balls to create four containers, as shown in the diagram below.

   |○○○○|○|○○○|

   There are seven places to put those walls. Since we are dealing with ordered triplets, we see that order does matter, and our method does take this into account. Also, there exists a direct analogy between counting placements of walls and counting ordered triplets of numbers that sum to 8. Hence the number of ways to place the two walls is

   $$\binom{7}{2} = 21$$

8. Consider the following diagram.

   Note that $OM \perp AE$ and $AM = ME$. Hence $AM = \frac{1}{2}$. By the Pythagorean Theorem, $OM = \sqrt{s^2 - \frac{1}{4}}$. But since $C, O, M$ are collinear, we know that $CM = CO + OM$. We also know that $AO = CO$. Hence $CM = s + \sqrt{s^2 - \frac{1}{4}}$. Using the Pythagorean Theorem again, we find that

   $$AC = \sqrt{2s^2 + 2s\sqrt{s^2 - \frac{1}{4}}}$$

   Equivalent answers, if not fully simplified, were also accepted with full credit.

9. Let $C$ have side length $s$. $T$ is a regular octahedron. Thus it can be broken down into two congruent square pyramids. The base of the square pyramids is a square with an area half that of a side of C, so $b = \frac{1}{2}s^2$. The height of the pyramids is half the length of a side of $C$, so $h = \frac{s}{2}$. Thus the volume of one of the pyramids is

   $$V = \frac{1}{3}bh = \frac{1}{3}\frac{1}{2}s^2\frac{1}{2}s = \frac{s^3}{12}$$

Thus the volume of $T$ is $\frac{s^3}{6}$. But the volume of $C$ is $s^3$, hence the desired ratio is

$$\frac{[T]}{[C]} = \frac{\frac{s^3}{6}}{s^3} = \frac{1}{6}$$

10. We can easily see that the smallest integer that is not a Fibonacci number is 4, which can be summed as $F_0 + F_3 = 1 + 3 = 4$. We next try to find the smallest number that cannot be the sum of at most two Fibonacci numbers. We have

$$\begin{aligned}
1 &= 1 \\
2 &= 2 \\
3 &= 3 \\
4 &= 3 + 1 \\
5 &= 5 \\
6 &= 5 + 1 \\
7 &= 5 + 2 \\
8 &= 8 \\
9 &= 8 + 1 \\
10 &= 8 + 2 \\
11 &= 8 + 3
\end{aligned}$$

But 12 cannot be represented as the sum of two Fibonacci numbers. We also note that $12 = 8 + 4$. 8 is the largest Fibonacci number that we encountered, and 4 is the smallest number that is not a Fibonacci number. We look for this pattern when trying to find our desired number.

We have $8 + 12 = 20 = 13 + 5 + 2$, so 20 doesn't work. Similarly, $13 + 12 = 25 = 21 + 3 + 1$. But consider $21 + 12 = 33$. Since $\frac{33}{3} = 11$, one of the Fibonacci numbers we use must be at least 11. Hence we must use 13 or 21. But we know that 21 won't work, because 12 isn't the sum of any two Fibonacci numbers. Similarly, 13 doesn't work, because $13 + 21 > 33$, and $33 - 13 - 13 = 7$ (which is not a Fibonacci number). Hence the desired number is 33.

Students could also find sums for all numbers less than 33, and then fail to find a sum for 33. However, our more logical approach above helps prove rigorously that 33 is the least such number.