NEMO 2008 - Tiebreaker Round

Solutions

1. Note that there are $\left\lfloor \frac{100}{3^x} \right\rfloor$ multiples of 3^x (for x a positive integer) less than or equal to 100. Hence there are 33 multiples of 3, 11 multiples of 9, 3 multiples of 27, and 1 multiple of 81. Hence $k = 33 + 11 + 3 + 1 = 48$. This holds because adding the number of multiples of 9 adds the extra power of 3 that each multiple of 9 contributes. The same goes for adding the number of multiples of 27 and 81. Students could also use the formula

$$k = \sum_{i=1}^{\lfloor \log_p n \rfloor} \left\lfloor \frac{n}{p^i} \right\rfloor$$

2. No such triangles exist with perimeters 1 or 2, and one exists with perimeter 3 (an equilateral triangle). If $p = 4$, then the sides must be $1, 1, 2$. But this violates the triangle inequality. Hence there are no triangles with perimeter 4. If $p = 5$, then we have side lengths of either $1, 2, 2$ or $1, 1, 3$. The latter violates the triangle inequality, hence there is one triangle with perimeter 5. If $p = 6$, we have $1, 1, 4$, $1, 2, 3$, or $2, 2, 2$. The first two violate the triangle inequality, hence there is one triangle with perimeter 5. If $p = 7$, we have $1, 1, 5$, $1, 2, 4$, $1, 3, 3$, or $2, 2, 3$. The first two violate the triangle inequality, hence there are two triangles with perimeter 7. The results are condensed in our table below.

Perimeter	1	2	3	4	5	6	7
Triangles	0	0	1	0	1	1	2

Hence there are five such triangles.

3. Let the equilateral triangle be ABC and let the center of the circles be O. Obviously, since $\triangle ABC$ is equilateral we have $\angle AOB = \angle COB = 30°$. Let M be the midpoint of BC. Obviously $OM \perp BC$, hence $\triangle OBM$ is a 30-60-90 triangle. Hence $2OM = OB$. But OB is the radius of the larger circle, and OM is the radius of the smaller circle. Hence the desired ratio is $\frac{1}{2}$.

4. Let $x = 1453$. Then the first product is equal to $x(x + 6)(x + 18) = x^3 + 24x^2 + 108x$. The second product is $(x - 2)(x + 10)(x + 16) = x^3 + 24x^2 + 108x - 320$. But we have

$$(x^3 + 24x^2 + 108x) - (x^3 + 24x^2 + 108x - 320) = 320$$

This value does not depend on x, so setting $x = 1453$ will not change the value.

5. There isn't a really neat trick for this one. Careful computation is the key.

We are given that $C_0 = C_1 = 1$. Hence $C_2 = 1 \cdot 1 + 1 \cdot 1 = 2$. $C_3 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$. $C_4 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14$. $C_5 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42$. $C_6 = 1 \cdot 42 + 1 \cdot 14 + 2 \cdot 5 + 5 \cdot 2 + 14 \cdot 1 + 42 \cdot 1 = 132$. $C_7 = 1 \cdot 132 + 1 \cdot 42 + 2 \cdot 14 + 5 \cdot 5 + 14 \cdot 2 + 42 \cdot 1 + 132 \cdot 1 = 429$. The results are summarized in the table below.

n	0	1	2	3	4	5	6	7
C_n	1	1	2	5	14	42	132	429

Hence $C_0 + C_1 + \cdots + C_7 = 1 + 1 + 2 + 5 + 14 + 42 + 132 + 429 = 626$. Students could attempt to collapse the recurrence expansions, but this did not significantly alter the amount of time required to solve the problem.

6. Let

$$\begin{aligned}
s_1 &= r_1 + r_2 + r_3 + r_4 + r_5 \\
s_2 &= r_1r_2 + r_1r_3 + r_1r_4 + r_1r_5 + r_2r_3 + r_2r_4 + r_2r_5 + r_3r_4 + r_3r_5 + r_4r_5 \\
s_3 &= r_1r_2r_3 + r_1r_2r_4 + r_1r_2r_5 + r_1r_3r_4 + r_1r_3r_5 + r_1r_4r_5 + r_2r_3r_4 + r_2r_3r_5 + r_2r_4r_5 + r_3r_4r_5 \\
s_4 &= r_1r_2r_3r_4 + r_1r_2r_3r_5 + r_1r_2r_4r_5 + r_1r_3r_4r_5 + r_2r_3r_4r_5 \\
s_5 &= r_1r_2r_3r_4r_5
\end{aligned}$$

The s_i are called the symmetric sums of the r_i. If we consider the polynomial

$$P(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5)$$

with some computation or knowledge of Vieta's formulas we can expand $P(x)$ as:

$$P(x) = x^5 - s_1x^4 + s_2x^3 - s_3x^3 + s_4x - s_5$$

Thus, for $P(x) = x^5 + 2x^4 - 5x^3 - 4x^2 + 4x + 1$, we find that

$$s_1 = -2, s_2 = -5, s_3 = 4, s_4 = -4, s_5 = -1$$

Now, we want to express $r_1^3 + r_2^3 + r_3^3 + r_4^3 + r_5^3$ in terms of the s_i. In order to get cubes of the roots, we will have to have a s_1^3 term in our sum. But this generates many excess terms. With a little manipulation, students can find that $s_1^3 - 3s_1s_2 + 3s_3 = r_1^3 + r_2^3 + r_3^3 + r_4^3 + r_5^3$. Hence

$$r_1^3 + r_2^3 + r_3^3 + r_4^3 + r_5^3 = s_1^3 - 3s_1s_2 + 3s_3 = (-2)^3 - 3 \cdot (-2) \cdot (-5) + 3 \cdot 4 = -8 - 30 + 4 = -34$$

7. Let the length of a side of the triangle be s, and let $\frac{PM}{AB} = \lambda$. Let N be the midpoint of AB. We know that $PN = (1 - \lambda)s$. Also, $PA = PB = \lambda s$. Because of the definition of N, we have $AN = BN = \frac{s}{2}$. Hence by the Pythagorean Theorem,

$$\frac{s^2}{4} + (1 - \lambda)^2 s^2 = \lambda^2 s^2$$

We first divide through by s^2. Then we solve the resulting equation for λ.

$$\begin{aligned}
\frac{1}{4} + (1 - \lambda)^2 &= \lambda^2 \\
\frac{1}{4} + (1 - 2\lambda + \lambda^2) &= \lambda^2 \\
\frac{5}{4} &= 2\lambda \\
\frac{5}{8} &= \lambda
\end{aligned}$$

Let $[XYZ]$ denote the area of the polygon XYZ. Since PAB is a triangle, we have

$$[PAB] = \frac{1}{2} PN \cdot AB = \frac{1}{2} \cdot \frac{3}{8} s \cdot s = \frac{3}{16} s^2$$

Since $PADM$ is a trapezoid, we have

$$[PADM] = \frac{1}{2} DM \cdot (PM + AD) = \frac{1}{2} \cdot \frac{1}{2} s \cdot \left(\frac{5}{8} s + s \right) = \frac{1}{4} s \cdot \frac{13}{8} s = \frac{13}{32} s^2$$

Hence the desired ratio is

$$\frac{[PAB]}{[PADM]} = \frac{\frac{3}{16} s^2}{\frac{13}{32} s^2} = \frac{3 \cdot 32}{13 \cdot 16} = \frac{6}{13}$$